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Logical Reformulation of Quantum Mechanics. II. Interferences and the Einstein–Podolsky–Rosen Experiment

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The consistent quantum interpretations of logic that were introduced in a previous paper are applied to four experiments: (1) ordinary interferences, (2) the Badurek–Rauch–Tuppinger neutron interferometry experiment, (3) the Einstein–Podolsky–Rosen experiment, and (4) the detection far away of the origin of a nonrelativistic particle initially near the origin. In the first two cases, the proposition calculus excludes the possibility of observing interferences and of asserting together through which path the particles went. It is used to provide a somewhat complete discussion of the Badurek–Rauch–Tuppinger experiment. The possibility of using logical implication allows a rather complete discussion of the EPR experiment, including the question of causality, although the lack of a relativistic version of the theory does not allow a complete discussion of causality. The last experiment leads to the following result: Detecting the position of a particle at time t sometimes allows one to determine with a finite uncertainty what its momentum was just before the position measurement, even when it is infinitely precise.

KEY WORDS: Quantum mechanics; interferences; Einstein–Pdolsky–Rosen.

1. INTRODUCTION

In a previous paper[1] hereafter I, it has been shown how to construct representations (interpretations) of logic within the framework of quantum mechanics. It will be assumed in the following that the basic ideas proposed in that paper can be used with no further comment and the corresponding notions will be freely used.

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The present article deals with four applications:

1. A discussion of a typical interference experiment. It is shown how the predicate "A particle went through a given hole" can never enter a consistent representation of logic along with the predicates associated with the observation of interferences. The first predicate is therefore meaningless according to the definition given in I.

2. A recent and more refined neutron interference experiment performed by Badurek *et al.*[2] where spin-flippers having different resonance frequencies act on two polarized neutrons beams is also considered. It is shown that, here again, one cannot logically tell through which beam the neutron went. This is related to the fact that the resonant magnetic fields are produced by coils where a classical current flows so that their quantum states are coherent in the sense of quantum electrodynamics.

3. The Einstein–Podolsky–Rosen experiment.[3] The intuitive reasoning that is frequently made to interpret it is given a rigorous logical basis.

4. The knowledge about the momentum before measurement that can be derived from a position measurement.

Some of these questions were first considered by Griffiths[12] in his original formulation of the theory and there is general agreement between the present results and his, except of course for what is peculiar to the use of implication.

2. INTERFERENCES

My first example will deal with interference experiments. For definiteness, I shall consider the case of Young's experiment: A particle with given momentum hits a screen pierced by two holes H_1 and H'_1. Behind the screen, a battery of detectors registers the particle.

In conventional measurement theory, it is known that, when detectors are located within the holes, wave packet reduction implies that no interference can be observed. However, when no such actual detector is present, the question has always been felt to be a delicate one. Ordinary intuition keeps telling us that the particle *had to* go through only one hole. The question to be examined is therefore whether this naive classical statement can be given a meaning within the framework of *some* consistent quantum representation of logic.

It has been shown in I that such a representation relies on a given initial state $|\psi\rangle = \exp(ikz)$ and a set of propositions. The predicates of interest are

$E_1(t_1)$: "the particle went through hole H_1 at time t_1"

$E_1'(t_1)$: "the particle went through hole H_1' at time t_1"

$E_2^n(t_2)$: "the particle hit a detector in position n at time t_2," with $n = 1, 2, 3, \ldots$.

If it can be asserted that the particle went through a given hole, then there should exist a consistent representation of logic containing all these predicates, at least for some choice of times $(t_1; t_2)$.

One might associate predicate E_1 with a three-dimensional region surrounding the first hole. However, this would lead to unnecessary mathematical developments that have nothing to do with the basic question. Accordingly, I shall slightly modify the experimental setup by introducing a horizontal plate along the x–z plane (see Fig. 1), which will not essentially modify the problem. Then E_1 will mean that the particle is on the left of the screen in the region $-a < z < a$, above the plate $(y > 0)$. E_1' will mean that it is in the region $-a < z < a$, $y < 0$, both predicates being taken at the same time t_1. The z coordinate of the screen in the chosen system of reference will be called L and I take $L > a$.

To make the predicates E_2^n explicit, I assume that a plane parallel to the screen has been covered by a battery of identical receptors so that E_2^n means that the particle is detected at time t_2 on detector number n located in a region of space having a volume V and centered at point x_n. The point x_0 will be taken on the z axis.

The initial wave function ψ on the left of the screen is taken as $K\exp(ikz)$, where K is a normalizing factor: $K = \Omega^{-1/2}$, where Ω is a large quantization volume. The corresponding density operator is $\rho = |\psi\rangle\langle\psi|$.

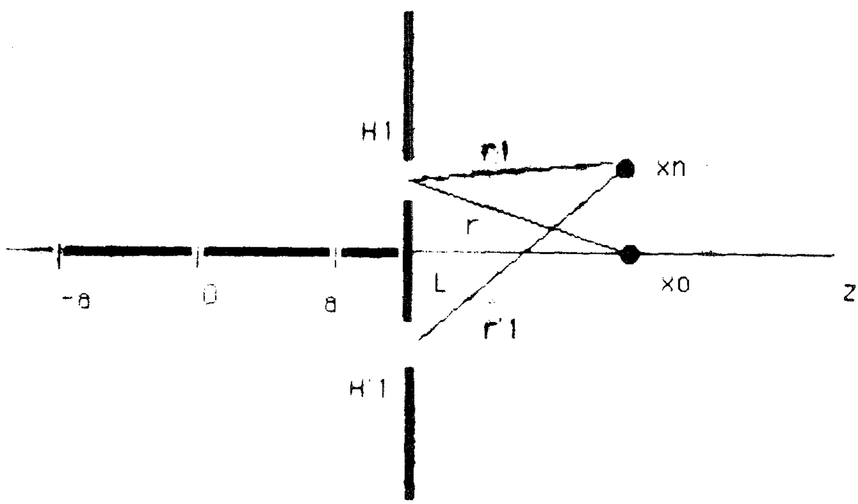

Fig. 1. The geometry of an interference experiment as discussed in the text.

I retain only the smallest representation of logic allowing these predicates because I want to prove them to the inconsistent. If the smallest representation containing them is inconsistent, the same will obviously be true for any large one.

The compatibility conditions take the form

$$C = \operatorname{Re} \operatorname{Tr}[E_1(t_1)\, \rho E_1'(t_1)\, E_2^n(t_2)] = 0 \tag{2.1}$$

As usual in the elementary treatment of such problems, I use the BKW approximation, amounting here to a replacement of the Schrödinger equation by the Huygens–Fresnel principle.

The projector E_1 can be written explicitly as

$$E_1 = \int_{-a}^{+a} dz \int_0^{\infty} dy \int_{-\infty}^{+\infty} dx\, |x, y, z\rangle\langle x, y, z| \tag{2.2}$$

One can pass to the Heisenberg representation, using the free-particle Green's function:

$$\langle x|\exp(-iHt)\,|y\rangle = (-2\pi i t)^{-3/2} \exp[i(x-y)^2/2t] \tag{2.3}$$

where I use for simplicity units where h and the particle mass are equal to unity.

Take the point of observation x^n not too far from the z axis, call r_1 and r_1' its distance, respectively, to the centers of the holes H_1 and H_1' and r its distance to the central point x_0. Also call T the probability amplitude for a particle to pass through any hole. Finally, choose the time difference $t_2 - t_1$ to be the classical time necessary to go from the plane of abscissa zero to x_0, i.e.,

$$t_2 - t_1 = (L + r)/v, \qquad v = k/m = k$$

With these assumptions, the trace in Eq. (2.1) takes the simple form

$$\langle x_n t_2|\, E_1(t_1)\,|\psi\rangle\langle\psi|\, E_1'(t_1)|x_n t_2\rangle\, V \tag{2.4}$$

The first factor in expression (2.4) can be written as

$$\int_{-\infty}^{+\infty} dx \int_0^{\infty} dy \int_{-a}^{+a} dz\, \langle x_n t_2|x t_1\rangle\langle x t_1|\psi\rangle \tag{2.5}$$

or, with our simplifying assumptions and putting $\Delta t = t_2 - t_1$, $\varepsilon = k^2/2$,

$$\Omega^{-1} \int_{-a}^{+a} dz\, (T/r_1)(-2\pi i\, \Delta t)^{-1/2} \exp[i(L + r_1 - z)^2/2\Delta t] \exp(ikz - i\varepsilon t_1) \tag{2.6}$$

Here appears for the first time a method that will be met several times in the sequel: The integral over z in Eq. (2.6) can be evaluated by the saddle-point method because all the quantities are essentially classical and the exponents contain a factor $\hbar^{-1}$ when ordinary units are reestablished. This means that the exponentials are very rapidly oscillating quantities, so that the asymptotic method of saddle-point (or stationary phase) integration can work. The integral (2.6) has the form

$$\int_{-a}^{+a} (-2\pi i \Delta t)^{-1/2} \exp(i\Phi)\, dz$$

with

$$\Phi = (L + r_1 - z)^2/2\Delta t + kz$$

has a saddle-point occurring at the solution of $\partial\Phi/\partial z = 0$, or $L + r_1 - z = v\Delta t$, i.e., $z_0 = r_1 - r$, where the exponent is $\Phi_0 = k(L + r_1)$. Since this saddle point lies well within the integration domain $[-a, +a]$, by performing a similar calculation for the second factor in (2.4) and assimilating r_1 and r_1' to r in the overall factors like r_1^{-1} in (2.6), one gets for the right-hand side of the compatibility condition (2.1) the result

$$\begin{aligned} C &= \mathrm{Re}\{V/(\Omega r^2)\,|T|^2 \exp[i(\phi - \phi')]\} \\ &= |T/r^2|^2 (V/\Omega) \cos(\phi - \phi') \end{aligned} \tag{2.7}$$

with $\phi = kr_1$, $\phi' = kr_1'$.

On the other hand, an easy trace calculation gives for the probability of the predicate E_2^n the familiar quantity

$$\begin{aligned} w_n &= |T/r|^2 (V/\Omega)\, |\exp(i\phi) + \exp(i\phi')|^2 \\ &= |T/r^2|^2 (V/\Omega)\, 2[1 + \cos(\phi - \phi')] \end{aligned} \tag{2.8}$$

Clearly C is not zero, so that the statement "The particle went through hole H_1" together with the observation at x_n cannot be given a meaning in any consistent quantum representation of logic. There is an exceptional case where C vanishes when the phase difference is equal to $\pi/2$ modulo π. However, since one must consider all the compatibility conditions for all the detectors, this is of no consequence.

Finally, it can be said that the statement "The particle went through one hole" is a meaningless naive statement. It only came from an intuition that was built ultimately upon our classical representation of logic that does not apply here.

This simple example shows how a proposition calculus can replace elusive arguments based upon wave packet reduction by unambiguous results from straightforward calculations. The answer proposed here to this question is not the usual one: “No measuring apparatus can tell because it would change the experimental setup.” It is not, “One can never tell, but maybe, who knows if there were hidden variables?” It is: “If its obeying a logic is included in the laws of Nature and if this is the type of logic it obeys, then the question is intrinsically meaningless.”

3. A NEUTRON INTERFEROMETRY EXPERIMENT

An interesting alternative to Young's experiment was proposed by Vigier[4] and was performed in a beautiful experiment by Badurek *et al.*[2] I describe it briefly.

A neutron beam is polarized along its velocity. It is separated by diffraction on a crystal into two well-separated beams 1 and 2 (see Fig. 2). Each beam passes through a spin-flipper, i.e., a magnetic resonance device composed of a coil generating an oscillating magnetic field, everything being in a stationary homogeneous magnetic field. The time of traversal of the spin-flipper by the neutron is such that the polarization should completely flip. Furthermore, an analyzer located behind the spin-flipper lets only spin-flipped neutrons pass. Finally, the two beams are recombined again by diffraction to let them interfere. The experimental setup is such that neutrons cross the device one by one.

One of the most interesting features of the experiment is that the alternating magnetic fields produced by the two coils have sharply separated resonance frequencies ω_1 and ω_2. Is this to say, as Vigier originally suggested, that we can tell through which beam the neutron went by detecting the energy of the photon that was either emited or absorbed during the spin flip? Surely, during a spin flip occurring in an external stationary magnetic field B, the energy of the neutron must change by a quantity $\frac{1}{2}gB$ that can be either $h\nu_1$ or $h\nu_2$ (g being the gyromagnetic

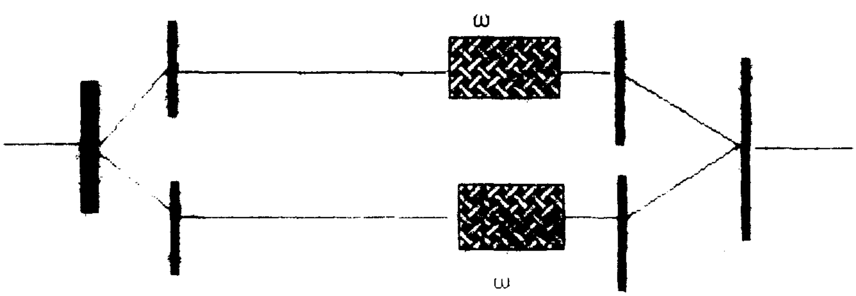

Fig. 2. Sketch of the Badurek–Rauch–Tuppinger experimental setup.

ratio). This energy must come from a photon and if we can tell its frequency, we can tell what beam the neutron followed.

I first consider the problem in conventional quantum mechanics. Since photons are involved, some reliance on quantum electrodynamics is necessary.

The classical magnetic field produced by the first coil has a vector potential that will be written as $\mathrm{Re}[\alpha A_1(x)\exp(-i\omega_1 t)]$ in the gauge $\mathrm{div}\, A = 0$. I shall consider its intensity α as a variable parameter, A_1 being normalized by

$$\int (A_1)^2\, dx = 1$$

In order to describe the associated photons, one may proceed as follows. Define a complete basis of orthonormal transverse vectorial functions, the first term in the basis being A_1. This is easily realized using Schmidt's orthogonalization procedure starting from an overcomplete family made from A_1 and a complete orthogonal basis. Then the quantized field can be expanded along this basis. In the present case, only photons corresponding to the first basis vector will enter into play, so that the relevant quantum field can be simply written in terms of creation/annihilation operators as

$$\begin{aligned} A(x, t) = {} & (\sqrt{2\omega_1})^{-1/2}[a_1^\dagger A_1(x)\exp(-i\omega_1 t) + a_1 A_1^*(x)\exp(i\omega_1 t)] \\ & + (\sqrt{2\omega_2})^{-1/2}[a_2^\dagger A_2(x)\exp(-i\omega_2 t) + a_2 A_2^*(x)\exp(i\omega_2 t)] \end{aligned} \tag{3.1}$$

where I have also introduced the field produced by the second coil.

The quantized field generated by the coil will be a coherent state since the *ac* currents producing them are essentially classical. Denoting by $|0\rangle$ the vacuum photon state, the state of the electromagnetic field will therefore be simply[5]

$$|\Phi\rangle = \exp(a_1^\dagger \alpha_1 + a_2^\dagger \alpha_2 - \tfrac{1}{2}|\alpha_1|^2 - \tfrac{1}{2}|\alpha_2|^2)|0\rangle \tag{3.2}$$

where α_1 and α_2 are c-numbers depending only upon the current intensities in the coils.

Assuming for simplicity that there is absorption of a photon in the spin-flipper (the case of emission would only lengthen the equations without any new physics), the state of the complete quantum state neutron + photons going out of the spin-flipper can easily be written: Let $\psi_{1-}(x)$ be the wave function of the spin-flipped neutron in beam 1 (its kinetic energy having increased by $h\nu_1$). With a similar notation for beam 2, the complete state vector reads

$$|\Psi(x)\rangle = [a_1\psi_{1-}(x) + a_2\psi_{2-}(x)]\,|\Phi\rangle \tag{3.3}$$

The probability density for observing the neutron somewhere behind the spin-flippers will be given by $\langle \Psi(x) | \Psi(x) \rangle$. However, coherent states satisfy the relation

$$a_{1,2} | \Phi \rangle = \alpha_{1,2} | \Phi \rangle \tag{3.4}$$

Therefore one has

$$\langle \Psi(x) | \Psi(x) \rangle = | \Psi_1(x) \alpha_1 + \psi_2(x) \alpha_2 |^2 \tag{3.5}$$

Accordingly, it is found that the interferences are observed when the electromagnetic field is in a coherent state, this result being essentially independent of the current intensity in the coils. The uncertainty relation between the phase and the number of photons has therefore nothing to do with it.

Can we tell which beam the neutron followed? In conventional quantum mechanics, this would mean detecting the fact that the electronic equipment delivered at some instant an excess energy corresponding to the energy of one photon, either in one coil or in the other. This is certainly impossible in practice, but is it inconceivable? The discussion could go on for a long time through the exchange of subtle "naive" arguments.

The solution I propose is simply to go back to the discussion given in Section 2. The only practical change will be to use state vectors of type (2.3) in a larger Hilbert space allowing for photons. Putting

$$| \Phi_1 \rangle = a_1 | \Phi \rangle = \alpha_1 | \Phi \rangle, \qquad | \Phi_2 \rangle = a_2 | \Phi \rangle = \alpha_2 | \Phi \rangle$$

the first term in Eq. (2.4) will remain essentially the same, except that one has to take into account the exchange of energy in the spin-flipper, as explained by Badurek *et al.*, and to introduce a multiplicative factor coming from the action of the annihilation operators in Fock space given by

$$\langle \Phi | a_{1,2} | \Phi \rangle = \alpha_{1,2}$$

Everything boils down, up to irrelevant normalization factors, to the replacement of C (the right-hand-side of the compatibility condition) and P_n (the probability of observation), respectively, by

$$\begin{aligned} C' &= \alpha_1 \alpha_2 \cos(\phi - \phi') \\ P_n &= |\alpha_1 \exp(i\phi) + \alpha_2 \exp(i\phi')|^2 \end{aligned} \tag{3.6}$$

The conclusions that were previously obtained remain unchanged: to tell which way the neutron went is still a logically meaningless statement.

For the discussion to be reasonably complete, two remarks should be added.

1. In a different experimental setup where the neutron beams would cross two masers located in a constant magnetic field, so that spin flip could occur by photon absorption and only one spin state be selected by sing a polarization filter, the analysis is similar but the results are somewhat different. One assumes that the state of the electromagnetic field in each maser (with frequencies ω_1 and ω_2) have a well-defined number of photons of each type N_1 and N_2. Then, reconsidering the analysis given in Section 2, one can replace the predicate E_1 by "There are $N_1 - 1$ photons of the first type and N_2 of the second type in the electromagnetic field at a time t_1 after crossing of the masers by the neutron." The predicate E_1' would be similar, exchanging the roles of N_1 and N_2. Then the orthogonality of photon states having different occupation numbers implies immediately that these predicates are *compatible* with the predicates E_2^n corresponding to observation. However, the probability for observing the neutron irrespective of any predicate becomes

$$\begin{aligned} &\langle N_1 N_2 | (\psi_{1-}^* a_1^\dagger + \psi_{2-}^* a^\dagger)(\psi_1 a_1 + \psi_2 a_2) | N_1 N_2 \rangle \\ &\quad = N_1 |\psi_{1-}|^2 + N_2 |\psi_{2-}|^2 \end{aligned} \tag{3.7}$$

up to an overall factor.

One can then tell which way the neutron went, but there are no interferences.

2. Since logic plays such a role in the present considerations, it may be worth mentioning that the apparently simple statement "a photon with frequency ν_1 was absorbed" that was important in Vigier's formulation[4] has in general no meaning because no projector can be associated to it. This is a theorem that is not too difficult to prove using the Bargman representation of Fock space,[6] but for which the proof will be omitted here. Of course, the predicate "There are N photons" has an associated projector.

4. THE EINSTEIN–PODOLSKY–ROSEN EXPERIMENT

The Eistein–Podolsky–Rosen experiment has had a great influence on reaserch dealing with the foundations of quantum mechanics. It was first proposed as a *gedanken* experiment[3] before being realized in the laboratory.[8]

One version is the following. An unstable particle Δ is confined near the origin. It can decay into two particles P and P' and there is no loss of generality in considering the case where P and P' have the same mass m. Particle P is observed at time t by a detector located far away from the origin, around a point x.

When the decaying particle decays into two spin-one-half particles in a total spin state zero and when the first particle P is observed to have a spin component $+1/2$ along a z axis, can we prove rigorously that the other particle has a z spin component $-1/2$? What happens when one wants to measure the spin of the second particle along a different axis? This question was considered by Bell in the framework of hidden-variables theory[9] and has since attracted much attention.

This problem has been considered by Griffiths,[12] so that I shall only state what differs in the present analysis from the one he gave, i.e., the use of implications. The calculations are essentially the same as his, except that I augment them by the results in measurement theory that were given in paper I.

I only consider spin in the present section. I shall introduce the following predicates:

F_1 = "A measuring apparatus gives a reading corresponding to the value $+1/2$ for the spin component of P along a direction n_1 at time t_1"

E_1 = "The spin of P along n_1 is $+1/2$ at time t_1"

E'_1 = "The spin of the other particle P' along n_1 is $-1/2$ at time t_1"

F_2 = "A second measuring apparatus gives a reading coresponding to the value $-1/2$ for the spin of the other particle P' along a direction n_2 at time t_2"

E_2 = "The spin of P' along n_2 is $+1/2$ at time t_2"

I take $t_2 \geqslant t_1$ for definiteness.

The results are the following:

(i) If n_1 is not parallel to n_2 and t_2 is later than t_1,

$$F_1 \Rightarrow E_1 \Rightarrow E'_1, \qquad F_2 \Rightarrow E_2$$

(ii) If n_1 is parallel to n_2 and t_2 is later than t_1,

$$F_1 \Rightarrow E_1 \Rightarrow E'_1 \Rightarrow E_2 \Rightarrow F_2$$

(iii) If n_1 is not parallel to n_2 and t_2 is equal to t_1, then E'_1 cannot enter a consistent representation of logic together with the other predicates. One has only

$$F_1 \Rightarrow E_1, \qquad F_2 \Rightarrow E_2$$

(iv) If n_1 is parallel to n_2 and t_2 is equal to t_1, then $E'_1 = E_2$ and, using the equivalence between two-sided implication and identity of two propositions,

$$F_1 \Rightarrow E_1 \Rightarrow E_2 \Rightarrow F_2, \qquad F_2 \Rightarrow E_2 \Rightarrow E_1 \Rightarrow F_1$$

so that $F_1 = F_2$.

The experiments are of course consistent with these statements.

In the language used at the end of the first paper, all these predicates can be said to be reliable.

It should be stressed that I am using here in fact nonrelativistic quantum mechanics, i.e., universal time. It would be very important to use, for instance, Dirac's equation to find out what these results become in a relativistic version. This will not be done here. However, I discuss in next section a few points concerning the spatial aspects of the experiment.

5. THE EPR EXPERIMENT: SPATIAL ASPECTS

I now discuss some aspects of the particles location in space at different times. This will proceed as follows. I first discuss the wave function of a system of two particles produced by the decay of a heavier one, and then discuss how the observation of one particle can be used to derive predicates concerning the other one. All the theory is nonrelativistic.

5.1. The Wave Function

I first discuss the wave function of two particles produced by a decay. At time zero, the unstable particle Δ is confined near the origin. It can be, for instance, an unstable nucleus in a crystal. It will be convenient to choose for its initial wave function a Gaussian state

$$\Phi(X) = (2\pi a^2)^{-3/4} \exp(-X^2/4a^2) \tag{5.1}$$

I denote its lifetime by τ and the corresponding width by $\Gamma = 1/\tau$. The rationalized Planck constant h will be taken equal to one.

Δ can decay into P and P'. Since even slight complications due to kinematics are of no concern here, I assume for simplicity that P and P' have the same mass m. Here again, the notation will be simplified by choosing m as the mass unit.

I call q the mean value of the momenta of P and P' in their center-of-mass system. In the present units, the corresponding nonrelativistic velocity is $v = q/m = q$. I treat P and P' as distinguishable, the case of undistinguishable particles offering no special difficulty.

The positions of P and P' will be denoted, respectively, by x and x'. The position of their center of mass and their relative separation will be denoted, respectively, by

$$X = (x + x')/2, \qquad \xi = x - x' \tag{5.2}$$

The wave function in the center-of mass system can be taken from Goldberger and Watson[11] as being

$$\phi(\xi; t) = (2\pi)^{-1}\sqrt{\Gamma/2\pi}(1/r)\int_0^\infty dk\,(k^2 - q^2 + i\Gamma/2)^{-1}\exp[i(kr - k^2 t)]$$
$$\times\{1 - \exp[i(k^2 - q^2 + i\Gamma/2)t]\} \tag{5.3}$$

where $r = |\xi|$.

It can be rewritten in a way exhibiting the time of decay t' as

$$\phi(\xi, t) = -i(2\pi r)^{-1}(\Gamma/2\pi)^{1/2}\int_0^\infty dk\,\exp(ikr)\int_0^t dt'$$
$$\times\exp[-ik^2(t - t') - iq^2 t' - \Gamma t'/2] \tag{5.4}$$

When the unstable particle decays, the center of mass becomes free and a^2 has to be replaced by $a^2 + i(t - t')/2$ to account for wave-packet spreading. However, the calculations can be slightly simplified if one considers a to be large enough so that $a^2 \gg (t - t')$. Note that this inequality is compatible with a large lifetime because it is not assumed that $a^2 \gg t$.

As shown by Golberger and Watson, if one denotes by H the total Hamiltonian including the decay Hamiltonian, the component of the wave function $\exp(-iHt)|\psi\rangle$ in the P–P' channel is then simply given by $\Phi(X)\,\phi(\xi, t)$.

It will be convenient to evaluate the wave function in the semiclassical approximation. To do so, one notices that the exponents in Eq. (5.4) contain in fact the Planck constant in the denominator. One can therefore evaluate first the integral over k by the stationary phase method. The stationary point corresponds to a value k_0 of k given by

$$r = 2k_0(t - t') \tag{5.5}$$

the stationary value of the exponent being given by

$$k_0 r - k_0^2(t - t') = r^2/4(t - t')$$

The integration over k gives rise to a factor $[8\pi i(t - t')]^{-1/2}$. The integration over t' can again be performed by the saddle-point method for the same reason. Neglecting the overall factors behaving as a power of $(t-t')$ for reasons to be checked later on, one finds the saddle point to be given at a value t'_0 for t' such that

$$r^2 = 4(q^2 - i\Gamma/2)(t-t')^2 \tag{5.6}$$

The small imaginary part resulting for t'_0 implies a slight displacement of the integration contour in the complex plane. The stationary value of the phase is then given by

$$r^2/4(t-t'_0) - (q^2 - i\Gamma/2)\,t'_0 = qr - q^2 t + i(\Gamma/2)(t - r/2q)$$

The integration over t' gives a factor $[8\pi i q^2/(t-t'_0)]^{-1/2}$, so that finally the wave function reads, up to an overall constant factor,

$$\psi(X, \xi, t) = \Phi(X)\,r^{-1}\exp[i(qr - q^2 t) - (\Gamma/2)(t - r/2q)] \tag{5.7}$$

The saddle-point approximation is correct if the stationary points are within the domain of integration, i.e., if

$$t - r/2v > 0 \tag{5.8}$$

expressing the fact that the decay happened at a positive time t'_0 and that the semiclassical travel of the particles occurred at velocity v.

Finally, it can be checked that the neglected factors containing powers of $(t-t')$ could be omitted if the confinement radius a is large enough so that

$$a^2 \gg r\lambda_{\mathrm{B}} \tag{5.9}$$

where $\lambda_{\mathrm{B}} = q^{-1}$ is the Broglie wavelength.

5.2. Global Spatial Analysis of the EPR Experiment

Let V be a small volume around a point x_0 where particle P is detected at time t. Let $E(t)$ denotes the predicate

$$E(t) = \text{"at time } t\text{, particle } P \text{ is in } V\text{"}$$

Let us now consider a sphere V' with a radius R to be made precise later on, centered at the symmetric point $-x_0$. Define the predicate

$$E'(t) = \text{"At time } t\text{, particle } P' \text{ is in } V'\text{"}$$

I now find under what conditions one has

$$E(t) \Rightarrow E'(t) \tag{5.10}$$

As was explained in I, one expects only to prove (5.10) up to negligible error.

Recall the strategy: One has to define a set X, a family of events B by giving its basis, to check the compatibility conditions, and finally to check that the conditional probability $P(E'^* \mid E)$ is very small.

Since one is dealing with particle positions, the spectra are respectively R^3 and R^3, so that $X = R^6$. As a basis of B, I take the following family:

$$(E \times E'),\ (E^* \times E'),\ (E \times E'^*),\ (E^* \times E'^*) \tag{5.11}$$

The compatibility conditions are trivial because all these projectors commute. The probabilities are all of the form

$$\mathrm{Tr}(\rho F F')$$

where ρ is the density operator $|\psi_{\Delta(t=0)}\rangle\langle\psi_\Delta(t=0)|$ and F represents either $E(t)$ or $E^*(t)$, F' representing $E'(t)$ or $E'^*(t)$. Since the volume V is very small, one has

$$w(E) = \mathrm{Tr}[\rho E(t)] = V \int_{R^3} dx' \, |\psi(x, x', t)|^2$$

$$w(E \cap E') = \mathrm{Tr}[\rho E(t)\, E'(t')] = V \int_{V'} dx' \, |\psi(x, x', t)|^2$$

$$w(E \cap E'^*) = \mathrm{Tr}[\rho E(t)\, E'^*(t)] = V \int_{V'^*} dx' \, |\psi(x, x', t)|^2$$

Assuming condition (5.8) to be satisfied, one can insert the wave function (5.7) into these integrals. It is the found immediately that the conditional probability $w(E'^* \mid E(= w(E \cap E'^*)/w(E)$ is of the order of $\exp(-R^2/2a^2)$, which may be taken as small as one wants by choosing R/a large enough. Thus one has obtained $E(t) \Rightarrow E'(t)$ u.t.n.e. The same result can be obtained using the wave function in the more precise forms (5.4) or (5.5). When wave-packet spreading is not negligible, the condition to be satisfied becomes $R^4 \gg a^4 + x_0^2 \lambda_B^2/4$.

5.3. The History of Particles

Let us now see how the logical reformulation of the interpretation of quantum mechanics can be used to assert what happened to a system at a time and at a place differing from the time and place where the measurement was performed. This is a typical example of violation of the "no trespassing" rules of the Copenhagen version. It is also an example showing how Griffiths' "consistent histories" can be substantiated.

It will not be convenient to work in three-dimensional space because this would lead to unnecessary mathematical complexity, so I shall assume here that the experiment takes place in a one-dimensional space. The calculations have also been done in three-dimensional space, but they turn out to be cumbersome and do not teach anything new.

Let us consider: (i) A very small space interval J_2 centered at a point x_{20} far from the origin. (ii) An interval J'_2 centered at the symmetric point $-x_{20}$ having a half-width R_2. Assume that the condition $R_2^2 \gg a^2$ is satisfied (a being, as above, the initial wave packet width). (iii) An interval J_1 centered at a point x_{10} located between the origin and x_{20}, with a half-width R_1 to be restricted later. (iv) An interval J'_1 with half-width R'_1 centered at $-x_{10}$ (see Fig. 3). Consider a time t_2 satisfying the condition (5.8), i.e., $t_2 > x_{20}/2v$, and a time t_1 such that $t_2 - t_1$ is the classical time spent by the particle when going from x_{10} to x_{20}, i.e.,

$$t_2 - t_1 = (x_{20} - x_{10})/v \tag{5.12}$$

Define the predicates

$E_1 =$ "At time t_1, P is in J_1"

$E'_1 =$ "At time t_1, P' is in J'_1"

$E_2 =$ "At time t_2, P is in J_2"

$E'_2 =$ "At time t_2, P' is in J'_2"

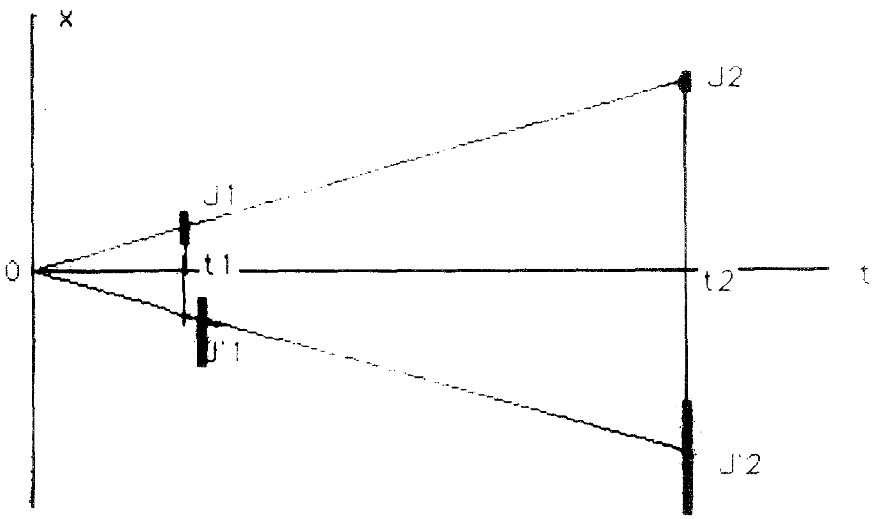

Fig. 3. The two-dimensional space-time regions used in the theory of histories and the discussion of causality.

The corresponding space X is R^4; the smallest family of events B has for basis the following sets

$$\begin{aligned} &D_1 = J_1 \times J_1' \times J_2' \times J_2, \qquad && D_2 = J_1 \times J_1'^* \times J_2' \times J_2 \\ &D_3 = J_1^* \times J_1' \times J_2' \times J_2, \qquad && D_4 = J_1^* \times J_1^{*\prime} \times J_2' \times J_2 \\ &D_5 = R \times R \times J_2^{*\prime} \times J_2, \qquad && D_6 = R \times R \times R \times J_2^* \end{aligned} \tag{5.13}$$

The corresponding compatibility conditions are

$$\begin{aligned} C_1 &= \operatorname{Re} \operatorname{Tr}(E_1 E_1' \rho E_1^* E_1' E_2' E_2) = 0 \\ C_2 &= \operatorname{Re} \operatorname{Tr}(E_1 E_1'^* \rho E_1^* E_1'^* E_2' E_2) = 0 \\ C_3 &= \operatorname{Re} \operatorname{Tr}(E_1 E_1' \rho E_1 E_1'^* E_2' E_2) = 0 \\ C_4 &= \operatorname{Re} \operatorname{Tr}(E_1^* E_1' \rho E_1^* E_1^{*\prime} E_2' E_2) = 0 \end{aligned} \tag{5.14}$$

The basic probabilities are $w(D_j)$, where $D_j = J_{1\alpha} \times J'_{1\beta} \times J'_{2\gamma} \times J_{2\delta}$ and where $J_{1\alpha}$, for instance, denotes either J_1, J_1^*, or the set of all real numbers R.

One already knows that E_2' is a logical consequence of E_2. One would like to find conditions under which E_1 and E_1' are also logical consequences of E_2, so that one might assert u.t.n.e. that these predicates describe a small part of the system history.

Writing, for instance,

$$E_1 = \int_{J_1} dx_1\, |x_1\rangle\langle x_1|, \qquad E_1(t_1) = U(t_1)\, E_1\, U^{-1}(t_1)$$

with

$$\langle x|\, U(t)\, |y\rangle = (-2\pi i t)^{-1/2} \exp[i(x-y)^2/2t] \tag{5.15}$$

one finds that the traces appearing in the compatibility conditions as well as the probabilities all take a similar form, namely that they are all proportional with a common factor to an integral such as

$$\int dx_1\, dx_1'\, dy_1\, dy_1'\, dx_2' \exp(i\Phi) \tag{5.16}$$

These integrals only differ by their domains of integration. For instance, in order to write the first consistency condition (5.14), one must integrate x_1 over J_1, x_1' over J_1', y_1 over $R - J_1 = J_1^*$, y_1' over J_1', and x_2' over J_2'.

The exponent Φ is given by

$$\begin{aligned} i\Phi = {} & i(x_{20}-x_1)^2/2(t_2-t_1) + i(x_2'-x_1')^2/2(t_2-t_1) - i(x_{20}-y_1)^2/2(t_2-t_1) \\ & - i(x_2'-y_1')^2/2(t_2-t_1) + iQ\,|x_1-x_1'| - iQ\,|y_1-y_1'| \\ & - \{[(x_1+x_1')/2]^2/4a^2\} - \{[(y_1+y_1')/2]^2/4a^2\} \end{aligned} \tag{5.17}$$

where I have used the one-dimensional version of the wave function (5.7) and where $Q = q - i\Gamma/4q$.

Before entering the calculation, let us first guess what the results should be. It was noticed in I that integrals like (5.16) were essentially the result of a Feynman path integral with a path starting from time zero, going to (x_2, t_2) and back, but restricted to specific "windows" at some intermediate times. One expects therefore to obtain always the same final result for all the integrals when the windows such as $(J_1, J_1', J_2', J_1, J_1')$ are crossed by the classical path of the system. Conversely, when one or several windows are not crossed by the classical trajectory, the resulting integral should be a small and oscillating quantity, the oscillation depending upon the exact position for the boundary of the windows. One therefore has to choose the boundaries of the intervals so as to satisfy the compatibility conditions.

Once again notice that the exponent Φ, when written in ordinary units, contains a factor h in the denominator: $x^2/2t$, for instance, should read $mx^2/2\hbar t$ and Qr should read $Qr/\hbar$.

I shall evaluate the integrals by the saddle-point method. Sometimes, it may look as if I am going to evaluate some of these integrals, when they are very small, by using the saddle-point method out of its range of validity, namely by evaluating the exponent by its second-order Taylor expansion far from the saddle point. This can be justified by taking explicitly into account the fact that the exponent is a second-order polynomial. Things are a bit complicated because of the absolute values appearing in them and I shall not give the necessary detailed analysis.

I make a Taylor expansion of Φ up to second order around its saddle point. The saddle point is the point where all the derivatives $\partial\Phi/\partial x_1$, $\partial\Phi/\partial x_1'$, $\partial\Phi/\partial y_1$, $\partial\Phi/\partial y_1'$ vanish. This gives for its coordinates $(x_{1s}, x_{1s}', y_{1s}, y_{1s}')$ the values

$$\begin{aligned} x_{1s} &= x_{10} - [2 + 16a^2/i(t_2-t_1)]^{-1}(x_2'+x_{20}) \\ x_{1s}' &= x_{1s} + [1 + i(t_2-t_1)/8a^2]^{-1}(x_2'+x_{20}) \end{aligned} \tag{5.18}$$

y_{1s} and y_{1s}' are given by similar expressions where i is to be replaced by $-i$.

Since the spreading of the wave packet is negligible, when x_2' is near the center $-x_{20}$ of J_2', this gives approximately

$$x_{1s} \approx y_{1s} \approx -x_{1s}' \approx -y_{1s}' \approx x_{10} \tag{5.19}$$

The stationary value of the exponent is equal to zero. In order to express the second-order terms, I define the variations

$$\xi = x_1 - x_{1s}, \qquad \xi' = x_1' - x_{1s}', \qquad \eta = {}_1 - y_{1s}, \qquad \eta' = y_1' - y_{1s}'$$

This gives for the Taylor expansion of Φ to second order:

$$\begin{aligned} i\Phi = &- [(\xi + \xi')^2/16a^2] - [(\eta + \eta')^2/16a^2] \\ &+ i(\xi^2 + \xi'^2 - \eta^2 - \eta'^2)[2(t_2 - t_1)]^{-1} \end{aligned} \tag{5.20}$$

When ξ' is integrated from $-\infty$ to $+\infty$, the resulting factor in ξ becomes

$$i\Phi(\xi) = \xi^2\{i[2(t_2 - t_1)]^{-1} - (16a^2)^{-1}\} \tag{5.21}$$

up to terms of higher order in $(t_2 - t_1)/16a^2$. Consider then the integral

$$I(R_1) = \int_{-R_1}^{R_1} \exp[i\Phi(\xi)]\, d\xi$$

Using the known asymptotic behavior of the error function, this is given, when R_1^2 is much larger than $(t_2 - t_1)/16a^2$, by

$$I(R_1) \approx \sqrt{\pi}\, \mathrm{Re}\{(\beta - i\alpha)^{-1/2} - \pi^{-1/2}[(\beta - i\alpha) R_1]^{-1} \exp[(i\alpha - \beta) R_1^2]\} \tag{5.22}$$

where $\beta = 1/16a^2$ and $\alpha = [2(t_2 - t_1)]^{-1}$.

With these notations, the four consistency conditions (5.14) become

$$\begin{aligned} C_1 &= \mathrm{Re}\{|I({}_1')|^2 I(R_1)[I(\infty) - I(R_1)]\} \\ C_2 &= \mathrm{Re}\{|I(\infty) - I(R_1')|^2 I(R_1)[I(\infty) - I(R_1)]\} \\ C_3 &= \mathrm{Re}\{|I(R_1)|^2 I(R_1')[I(\infty) - I(R_1')]\} \\ C_4 &= \mathrm{Re}\{|I(\infty) - I(R_1)|^2 I(R'_1)[I(\infty) - I(R_1')]\} \end{aligned} \tag{5.23}$$

Choosing R_1 and R'_1 to be large compared to $[\hbar/m(t_2 - t_1)]^{1/2}$ and a, one has essentially $I(R_1) = I(\infty)$, i.e., a finite constant, whereas Eq. (5.22) shows that $I(\infty) - I(R_1)$ is an exponentially small quantity oscillating with a wavelength of the order of $[\hbar/m(t_2 - t_1)]/R_1$ when R_1 changes from a given value R_1 to a neighboring one $R_1 + \Delta R_1$. Accordingly, one can choose the size of the intervals J_1 and J'_1 so that the consistency conditions are exactly satisfied.

The fact that the size of the intervals has to be chosen in such a precise way may be somewhat surprising. It was shown in I how this may be avoided by relaxing the axioms of probability theory. In the third paper of this series, it will also be shown how a more careful consideration of the meaning of projectors may clarify this point.

Introducing the small quantities

$$\varepsilon(R_1) = [I(\infty) - I(R_1)]/I(\infty) \approx [\hbar/m(t_2 - t_1)]^{1/2}/R_1$$

and $\varepsilon(R'_1)$ given by a similar expression, one can finally get the relevant probabilities. The integration of x'_2 over the interval J'_2 having the same half-width R'_2 as the sphere V' in Section 6 only gives a constant factor $2R'_2$. Including all the constant factors in $P(E_2)$, one gets

$$\begin{aligned} w(E_1, E'_1, E'_2, E_2) &= w(E_2) \\ w(E_1^*, E'_1, E'_2, E_2) &= w(E_2)\,\varepsilon^2(R_1) \\ w(E_1, E_1'^*, E'_2, E_2) &= w(E_2)\,\varepsilon^2(R'_1) \end{aligned} \tag{5.24}$$

According to the rules of implication given in I, it can be concluded that $E_2 \Rightarrow E'_2$ and $E'_1 \Rightarrow E_1$, u.t.n.e. There is no apparent difficulty, except for the heavier character of the calculations, to extend this result to many more predicates and therefore to logically derive a number of statements concerning the history of the system.

5.4. Causality

The EPR experiment can be considered from another standpoint. Observation tells us that particle P is in the interval J_2 at time t_2. Then, knowing that the dynamics obeys relativistic constraints, one can draw logical consequences of this event toward the past and toward the future. Let us now reconsider the above logical connection of several events in this new light.

The preceding analysis can be slightly extended in such a way that the family of events describing, at least partially, the history of the system consists of causally connected events, i.e., of timelike-separated events.

To obtain this result, it is enough to replace the time t_1 at which the position of particle P' was asserted by the predicate E'_1 by a slightly later time t'_1. One can thus realize a situation where the three pairs of events (E_2, E_1), (E_1, E'_1), and (E'_1, E'_2) are all made up of timelike-separated events. It is enough for that to take x_{10} much smaller than x_{20} and the space-time set (J'_1, t'_1) to be within the intersection of the light cones originating from the points of the set (J_1, t_1).

One can push the center x_{10} of the interval J_1 as near to the origin as one can, but not nearer than a distance remaining large compared to $[\hbar/m(t_2 - t_1)]^{1/2}$, and do the same for J'_1. The timelike separation will then be realized if one takes

$$t'_1 - t_1 \gg 2(|x_{20}|\,\lambda_B)^{1/2}/c$$

The calculations showing consistency and implication will be practically unchanged if the difference $t'_1 - t_1$ remains very small compared to $t_2 - t_1$, i.e., practically $|x_{20}|/v$. Both conditions can be realized together if one has

$$v/c \ll (|x_{20}|/\lambda_B)^{1/2}$$

an inequality easy to satisfy.

It is then possible to organize a chain of implications that is nowhere in conflict with the relativistic finite velocity of signals, namely: "Observing P in J_2 at time t_2" $\Rightarrow$ "P was already in J_1 at time t_1" $\Rightarrow$ "The other particle P' was in J'_1 at time t'_1" $\Rightarrow$ "P' is in J'_2 at time t_2." In fact, this is just the kind of reasoning that one makes "naively" when discussing the EPR experiment in ordinary terms.

The theory that has been given here could in principle be made relativistic but already in the present approximate treatment of the relativistic relations of events, it can be seen that the logical analysis cannot meet contradiction with the fact that signals have a finite velocity.

6. WHAT A POSITION MEASUREMENT TELLS US ABOUT MOMENTUM

I now consider a very common experiment that clearly shows how the logical interpretation of quantum mechanics differs from the Copenhagen formulation. It was already stressed that many statements that were strictly forbidden in the Copenhagen version become meaningful in the present one.

A typical example is the following. Suppose a particle is produced at the origin of space at time zero and that it is observed at a point x at time t. An intuitive naive statement would be to say that the velocity of the particle at any time before the measurement was x/t. This is said to have absolutely no meaning in the Copenhagen interpretation. I now show that it is nevertheless a reliable statement as defined in I.

I consider the case where the wave function at time zero is a Gaussian with uncertainty a in the position. Denoting the position by y, the wave function is therefore

$$\psi(y) = (2\pi a^2)^{-1/4} \exp[-(y^2/4a^2)] \tag{6.1}$$

Let us define the following predicates:

E_2 = "The particle is in an infinitely small volume ΔV around point x at time t"

E_1 = "The momentum of the particle is in a box B centered at the point $p_0 = mx/t$ at any time t_1 between zero and t"

The smallest representation of logic containing these predicates has the basis $\{(E_1, E_2), (E_1^*, E_2), (I_1, E_2^*)\}$. Choose a small number ε defining what we mean by a negligible error. In order to formalize the naive argument, one has to find whether this representation of logic is consistent and whether one has $E_2 \Rightarrow E_1$ up to negligible error of order ε.

We need the error function

$$\Phi(u) = (2/\sqrt{\pi}) \int_0^u \exp(-s^2)\, ds \tag{6.2}$$

as well as its asymptotic behavior for large arguments

$$\Phi(u) \approx 1 - (\pi u)^{-1} \exp(-u^2) \qquad (u \gg 1) \tag{6.3}$$

There is only one consistency condition, namely

$$\mathrm{Re}[\mathrm{Tr}(E_1 \rho E_1^* E_2)] = 0 \tag{6.4}$$

Introducing the wave function at time t in momentum space

$$\Psi(p, t) = (2a^2/\pi\hbar^2)^{1/4} \exp[-(p^2 a^2/h^2) - i(p^2 t/2m\hbar)] \tag{6.5}$$

the consistency condition reads

$$\mathrm{Re}\left\{\int_B dp \int_{B^*} dp'\, \Psi(p, t)\, \Psi^*(p', t) \exp[i(p' - p)x/\hbar]\right\} = 0 \tag{6.6}$$

and the probability of (E_1, E_2) is

$$w(E_1, E_2) = w(E_2) \int_B dp \int_B dp'\, \Psi(p, t)\, \Psi^*(p', t) \exp[i(p' - p)x/\hbar] \tag{6.7}$$

Take the box B to be a circular cylinder with axis along the direction of the point x with radius Δp and length $2\Delta p$. All the calculations are straightforward: the results of the integrations can be expressed as combinations of exponentials and error functions. In the interesting cases, the error functions can be approximated by their asymptotic expansion.

Introduce a time $\tau = (2Ma^2/\hbar)^{1/2}$ that is essentially the time necessary for a significant spreading of the initial wave packet. I consider the case where the time of observation t is large compared with τ. Moreover, I take

$$\Delta p \gg (m|x|\tau/t^2) \tag{6.8}$$

I say, as in paper I, that $E_2 \Rightarrow E_1$ up to error ε iff the conditional probability $P(E_1^* \mid E_2)$ is smaller than a fixed small number ε.

The consistency condition becomes in these conditions

$$\mathrm{Re}\{\exp[(-\Delta p^2(\tau + it)/m\hbar) - i\pi/4]\} = 0 \tag{6.9}$$

and the order of magnitude of the conditional probability is

$$P(E_1 \mid E_2) = O[(t\Delta p^2/m\hbar)^{-1/2} \exp(-a^2 \Delta p^2/\hbar^2)] \tag{6.10}$$

This last quantity is identified to the allowed error ε. It is seen that the error Δp thus defined is controlled by the Heisenberg uncertainty relation associated with the *initial* wave function. It expresses how uncertain is the naive statement expressing that the particle was initially at the origin of space.

In order to have $p_0 \gg \Delta p$ together with the condition (6.8), one must take x large enough so that

$$|x| \gg (\hbar t/ma)|\log \varepsilon| \tag{6.11}$$

i.e., the observation of the particle should have for its cause the occurrence of a large initial momentum and not just the spreading of the wave packet. Finally, it is easy to choose p in a precise way in order to satisfy the consistency condition, obtaining $E_2 \Rightarrow E_1$ up to negligible error of order ε.

It should be noticed that the intermediate values of the momentum are known with a high confidence that depends only upon the initial state and not on the accuracy of the position measurement. In fact, I have assumed the position measurement to be made with a very small uncertainty and find that the intermediate momentum is known with a confidence that is not limited by the Heisenberg uncertainty relation in the final measurement.

An analogous calculation can be done for an initial wave function $r^{-1} \exp(ikr - r/a)$, using saddle-point evaluations. The results are similar, but, when the position and time of the measurement are in the ratio $|x|/t$ of the order of hk/m, the reason for the observation can be assigned to the initial average value hk of the momentum and not to momenta having a small probability. This leads to a positive answer to a long-standing problem: when we observe a particle generated in an S wave, we can tell that the particle had its momentum directed toward the observation point some time before the measurement. The calculation can also be done for a particle produced in a decay, using the wave function given by Goldberger and Watson.

Here is a result that should obviously be investigated in full detail because it brings something somewhat new in quantum mechanics. It has some of the features that are necessary to satisfy the criteria put forward by Einstein *et al.*[3] as the conditions of "a complete description of physical reality" by quantum mechanics. It will be reconsidered in a later publication both in the EPR framework and for its own physical meaning.

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